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# Approximate Equations for the Flexure of Thin, Incomplete, Piezoelectric Bimorphs

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SUMMARY

In this paper the linear, three-dimensional, piezoelectric equations for a body in equilibrium are reduced to approximate, two-dimensional ones, treating the flexure of thin bimorphs, partly coated by electrodes (incomplete bimorphs). For that purpose two-dimensional equations are derived for piezoelectric plates and for bimorphs with completely coated faces. An assumption about the charge distribution on the inner electrode is given, stating that the charge vanishes on those parts where the outer faces are free of electrodes. This assumption allows the application of the mentioned, approximate equations for plates and bimorphs to the parts of incomplete bimorphs. By stating edge and continuity conditions, the approximate theory ls completed. The solution for a circular, incomplete, piezoceramic bimorph, loaded by a singular force in the centre, is given and compared with experimental results.

## 1. Introduction

A bimorph is a bilaminar disk, which can convert mechanical energy into electrical energy and vice versa. The bimorph considered here, consists of two piezoelectric plates, glued to each other with in between an infinitesimally thin electrode. The crystallographic axes of both plates have the same orientation. The outside faces are partly coated by shorted electrodes of identical shape, situated symmetrically with respect to the central electrode (fig. 1). The part of the bimorph covered by electrodes is denoted by region I and the remaining part by region II. We assume that the smallest lateral dimension of both regions are large with respect to the thickness $2h$. In the remainder these bimorphs are called incomplete bimorphs, to distinguish them from the ordinary bimorphs with completely coated faces (complete bimorphs).

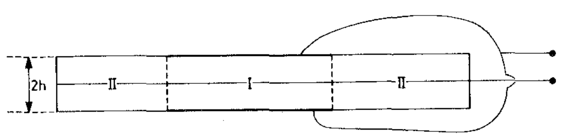


Figure 1. Cross section of the incomplete bimorph.

Piezoelectricity is a reversible, electro-mechanical phenomenon. In materials exhibiting this effect, stresses and strains occur when an electric field is applied and inversely, mechanical stresses produce electric polarization and hence an electric field. The piezoelectric phenomenon is described in many books, for instance [1], [2] and [3].

The bilaminar disks under consideration can be bent by transverse mechanical loads in the usual manner and by electrical charges on the electrodes. In the latter case one piezoelectric plate expands and the other one contracts. Since slipping is prevented, bending occurs. In this way bimorphs, vibrating in the quasistatic range of frequencies, are used in acoustical applications, such as loudspeakers, microphones and phonograph pick-ups.

In order to describe the behaviour of these quasistatic vibrations, the linear, three-dimensional piezoelectric relations for a generally anisotropic, homogeneous, piezoelectric body are reduced to approximate, two-dimensional ones, treating the static flexure of thin, incomplete bimorphs in vacuum. In the remainder we assume that the charge on the central electrode is only distributed on the part situated in region I. Hence the charge distribution on this electrode is neglected in region II. This assumption is suggested by the fact that in case a charged, half-infinite electrode is situated on a constant distance $h$ from an infinite, earthed one and the electrodes are placed in vacuum, the charge distribution on the infinite electrode vanishes in every fixed point of the protruding part as $h \to 0$ ([4]).

In virtue of the assumption introduced above, approximate equations can be stated for part I, being a complete bimorph, and part II. They are based on some assumptions which are derived from the exact solution for an unbounded bimorph, respectively plate. The two-dimensional theory for incomplete bimorphs is completed by giving appropriate conditions of continuity on the common boundary of the regions I and II and the edge conditions.

Finally the solution for a circular, piezoceramic, incomplete bimorph is given and experimental data are added. These results are in agreement with the theoretical solution.

## 2. Basic Equations

The points in the three-dimensional Euclidian space $L_{(3)}$ are referred to a set of coordinates $\theta^i$, $i=1, 2, 3$, in the usual manner. Unless specified, the coordinate lines are curvilinear.

The general equations are given in tensor notation. A summary of tensor calculus is given in [5] and [6]. A comma followed by an index $i$ denotes ordinary differentiation with respect to $\theta^i$ and a vertical line followed by an index covariant differentiation. Also the summation convention for repeated indices is employed. Latin indices range over 1, 2 and 3 and Greek indices over 1 and 2.

With respect to m.k.s. units we have the following, linear, three-dimensional relations for a piezoelectric body in equilibrium.

i) The piezoelectric equations; of the four equivalent sets of equations describing piezoelectricity two will be adopted,

$$T^{ij} = {}^{E}c^{ijkl} S_{kl} - e^{kij} E_k\,, \quad D^i = e^{ikl} S_{kl} + {}^{S}\varepsilon^{ik} E_k\,, \tag{2.1.a}$$

and $$S_{ij} = {}^{E}s_{ijkl} T^{kl} + d^{k}_{.ij} E_k\,, \quad D^i = d^{i}_{.kl} T^{kl} + {}^{T}\varepsilon^{ik} E_k\,. \tag{2.1.b}$$

ii) The equations of equilibrium,

$$T^{ij}|_i = 0\,. \tag{2.2}$$

iii) The strain displacement relations,

$$S_{kl} = \tfrac{1}{2}(U_k|_l + U_l|_k)\,. \tag{2.3}$$

iv) The Maxwell equations of electrostatics,

$$E_i = -V_{,i}\,, \tag{2.4.a}$$

$$D^i|_i = 0\,. \tag{2.4.b}$$

Body-forces, body-couples and surface-couples are neglected as usual in the theory of piezoelectricity ([7], [8]). The strains and stresses are denoted by the symmetrical tensors $S_{ij}$ and $T^{ij}$, the components of the electric field and electric displacement by $E_i$, respectively $D^i$, the components of the mechanical displacement by $U^i$ and the electric potential by the invariant function $V$. The tensors ${}^{E}s_{ijkl}$ and ${}^{E}c^{ijkl}$ represent the elastic coefficients, $e^{kij}$ and $d^{k}_{.ij}$ the piezoelectric constants and ${}^{S}\varepsilon^{ik}$ and ${}^{T}\varepsilon^{ik}$ the dielectric constants. The superscripts $E$, $T$ and $S$ denote that the coefficients are measured respectively at constant electric field, stress or strain. These coefficients satisfy the symmetry relations,

$${}^{E}c^{ijkl} = {}^{E}c^{jikl} = {}^{E}c^{klij}\,, \quad e^{kij} = e^{kji}\,, \quad {}^{S}\varepsilon^{ik} = {}^{S}\varepsilon^{ki}\,, \tag{2.5}$$

and analogous relations for ${}^{E}s_{ijkl}$, $d^{k}_{.ij}$ and ${}^{T}\varepsilon_{ik}$.

We now summarize the transition conditions of electrostatics*). Denoting the tangential

* These conditions are usually known as the boundary conditions. Since confusion is possible we prefer transition conditions.

components of the electric field and the normal components of the electric displacement on the boundaries of the dielectrics (1) and (2) by $E^{(1)}_{(t)}$, $E^{(2)}_{(t)}$, respectively by $D^{(1)}_{(n)}$ and $D^{(2)}_{(n)}$, we have

$$E^{(2)}_{(t)} - E^{(1)}_{(t)} = 0\,, \tag{2.6.a}$$

$$D^{(2)}_{(n)} - D^{(1)}_{(n)} = F. \tag{2.6.b}$$

The unit vector $\boldsymbol{n}$ is normal to the boundary and directed from dielectric (1) into dielectric (2). $F$ represents the free charge per unit area on the boundary surface. In (2.6) brackets are used in order to indicate that the indices are no tensor indices.

Finally we discuss the internal energy. When $S_{ij}$ and $D^i$ increase by $dS_{ij}$ respectively $dD^i$, the internal energy $A$ per unit volume stored in the body increases with the amount $dA$, given by ([1]),

$$dA = T^{ij} dS_{ij} + E_i dD^i\,, \tag{2.7}$$

neglecting contributions which are small of order $(dS_{ij})^2$ and $(dD^i)^2$.

We assume that $A$ vanishes whenever the stresses and the electric field vanish. Since $dA$ is a perfect differential ([1]), $A$ is independent of the manner in which we arrive at the final state. Hence $A$ is given by the expression

$$A = \int_0^1 (T^{ij} S_{ij} + E_i D^i) \lambda\, d\lambda \tag{2.8.a}$$

yielding,

$$A = \tfrac{1}{2}(T^{ij} S_{ij} + E_i D^i)\,. \tag{2.8.b}$$

In virtue of the piezoelectric relations, $A$ is a homogeneous quadratic function of six elastic and three electric quantities. As usual we assume that $A > 0$ for compatible, non-zero values of its arguments.

## 3. Bending of an Infinite Plate by Moments

An unbounded piezoelectric plate in vacuum is considered, bent by moments at infinity. Cartesian coordinates $(x_1, x_2, x_3)$ are chosen with $x_3 = \pm h$ defining the faces of the plate; $h$ is constant. In virtue of the choice of coordinates there is no difference between covariant and contravariant indices, hence only subscripts are used.

The stress distribution in the plate is assumed to be

$$T_{\alpha\beta} = \frac{3M_{\alpha\beta}}{2h^3} x_3, \qquad T_{j3} = 0\,, \tag{3.1}$$

where $M_{\alpha\beta}$ are the constant bending and twisting moments per unit of length. Also we assume the strains, electric field and electric displacement to depend only on $x_3$. Then it follows from (2.4) that $D_3$, $E_1$ and $E_2$ are constant throughout the plate. At the faces $x_3 = \pm h$ these quantities have to satisfy the transition conditions (2.6). Since no charges are present outside the body the external electric field vanishes. Hence we have inside the plate

$$D_3 = E_1 = E_2 = 0\,. \tag{3.2}$$

By substituting (3.1) and (3.2) into the piezoelectric relations (2.1.b), we obtain for the remaining mechanical and electrical variables,

$$S_{ij} = \frac{3M_{\alpha\beta}}{2h^3} s^{*}_{ij\alpha\beta} x_3\,, \tag{3.3.a}$$

$$D_\gamma = \frac{3M_{\alpha\beta}}{2h^3} d^{*}_{\gamma\alpha\beta} x_3\,, \tag{3.3.b}$$

$$E_3 = -\frac{3M_{\alpha\beta}d_{3\alpha\beta}}{2h^3\,{}^T\varepsilon_{33}}x_3\,, \tag{3.3.c}$$

where

$$s^*_{ij\alpha\beta} = {}^E s_{ij\alpha\beta} - \frac{d_{3ij}d_{3\alpha\beta}}{{}^T\varepsilon_{33}}\,, \tag{3.4.a}$$

$$d^*_{\gamma\alpha\beta} = d_{\gamma\alpha\beta} - \frac{d_{3\alpha\beta}\,{}^T\varepsilon_{\gamma 3}}{{}^T\varepsilon_{33}}\,. \tag{3.4.b}$$

Combining (3.3.a) and (2.3), the geometrical displacements can be evaluated. The formulae for the strains are analogous to the ones for an anisotropic elastic plate in a state of pure bending, only the coefficients now include the piezoelectric effect. In order to avoid rigid body motions we assume at $x_1 = x_2 = x_3 = 0$,

$$U_1 = U_2 = U_3 = 0\,, \tag{3.5.a}$$

$$U_{1,2} - U_{2,1} = U_{3,2} = 0\,. \tag{3.5.b}$$

Then the geometrical displacements become

$$U_\gamma = \frac{3M_{\alpha\beta}}{2h^3}s^*_{\gamma j\alpha\beta}x_j x_3\,, \tag{3.6.a}$$

$$U_3 = \frac{3M_{\alpha\beta}}{4h^3}\left(s^*_{33\alpha\beta}x_3{}^2 - s^*_{\gamma\delta\alpha\beta}x_\gamma x_\delta\right). \tag{3.6.b}$$

From (3.3.a) and (3.6.b) it follows

$$S_{\alpha\beta} = -U_{3,\alpha\beta}x_3\,. \tag{3.7}$$

The potential $V$ in the plate follows from (3.2) and (3.3.c). Defining $V(x_3 = h) = 0$, we obtain

$$V = -\frac{3M_{\alpha\beta}d_{3\alpha\beta}}{4h^3\,{}^T\varepsilon_{33}}(h^2 - x_3^2)\,. \tag{3.8}$$

Since all equations are satisfied exactly by the solution given above, this solution is exact.

Now we suppose that in case of arbitrary bending the stresses, strains, electric field and electric displacement do not vary substantially with respect to $x_1$ and $x_2$ over distances which are of order of the thickness of the plate. This enables us to use locally their distributions over the thickness as given by the exact solution of section 3. Hence effects of shear on flexure are neglected.

## 4. Two-Dimensional Equations for a Plate

Here the approximate two-dimensional equations for a thin piezoelectric plate in vacuum, possessing anisotropy of the most general form and bent by a transverse load, are given. The position vector $\boldsymbol{R}$ of a point of the plate will have the form ([5], page 185),

$$\boldsymbol{R} = \boldsymbol{r}(\theta^1, \theta^2) + \theta^3\boldsymbol{a}_3\,, \tag{4.1}$$

where $\theta^3 = 0$ represents the plane middle surface of the plate; $\boldsymbol{a}_3$ is a constant unit vector perpendicular to the middle surface. The faces of the plate are given by $\theta^3 = \pm h$ (fig. 2).

In virtue of the assumed local distribution of the stresses, the $T^{j3}$ vanish. Hence the strains $S_{j3}$ can be eliminated from the piezoelectric relations (2.1.a), yielding

$$T^{\alpha\beta} = c^{\alpha\beta\gamma\delta}_{(1)}S_{\gamma\delta} - e^{k\alpha\beta}_{(1)}E_k\,, \tag{4.2.a}$$

$$D^i = e^{i\gamma\delta}_{(1)}S_{\gamma\delta} + \varepsilon^{ik}_{(1)}E_k\,. \tag{4.2.b}$$

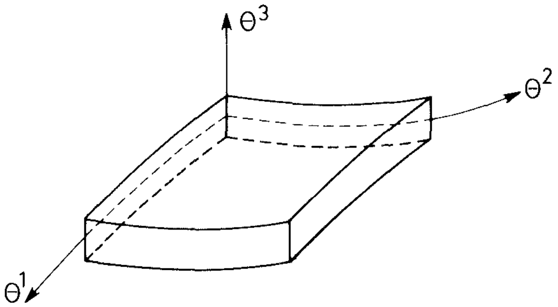


Figure 2. The coordinate curves.

$c_{(1)}^{\alpha\beta\gamma\delta}$, $e_{(1)}^{k\alpha\beta}$ and $\varepsilon_{(1)}^{ik}$ represent the elastic, piezoelectric and dielectric constants after elimination of the $S_{j3}$. From (3.7),

$$S_{\alpha\beta} = -W|_{\alpha\beta}\theta^3 , \tag{4.3}$$

where by definition $W = U_3\ (\theta^1, \theta^2, 0)$, hence the deflection of the middle plane. Substituting (4.3) and assumption (3.2) into (4.2), we obtain

$$T^{\alpha\beta} = -c_{(2)}^{\alpha\beta\gamma\delta} W|_{\gamma\delta}\theta^3 , \tag{4.4.a}$$

$$E_3 = \frac{e_{(1)}^{3\gamma\delta}}{\varepsilon_{(1)}^{33}} W|_{\gamma\delta}\theta^3 , \tag{4.4.b}$$

where

$$c_{(2)}^{\alpha\beta\gamma\delta} = c_{(1)}^{\alpha\beta\gamma\delta} + \frac{e_{(1)}^{3\alpha\beta} e_{(1)}^{3\gamma\delta}}{\varepsilon_{(1)}^{33}} . \tag{4.5}$$

The plate is bent by a transverse mechanical load of the form

$$\boldsymbol{q} = q\boldsymbol{a}_3 . \tag{4.6}$$

Then the integrated equations of equilibrium, following from (2.2), read,

$$M^{\alpha\beta}|_{\alpha} - Q^{\beta 3} = 0 , \tag{4.7.a}$$

$$Q^{\alpha 3}|_{\alpha} + q = 0 . \tag{4.7.b}$$

The $M^{\alpha\beta}$ and $Q^{\alpha 3}$ are defined by

$$M^{\alpha\beta} = \int_{-h}^{h} T^{\alpha\beta}\theta^3\, d\theta^3 , \tag{4.8.a}$$

and

$$Q^{\alpha 3} = \int_{-h}^{h} T^{\alpha 3}\, d\theta^3 . \tag{4.8.b}$$

Elimination of $Q^{13}$ and $Q^{23}$ from (4.7) yields

$$M^{\alpha\beta}|_{\alpha\beta} = -q . \tag{4.9}$$

Combining (4.4.a), (4.8) and (4.9) a fourth order partial differential equation for the deflection $W$ is obtained,

$$c_{(2)}^{\alpha\beta\gamma\delta} W|_{\alpha\beta\gamma\delta} = \frac{3q}{2h^3} . \tag{4.10}$$

The piezoelectric effect is included in the coefficients $c_{(2)}^{\alpha\beta\gamma\delta}$. When the piezoelectric effect is neglected, the $c_{(2)}^{\alpha\beta\gamma\delta}$ reduce to elastic coefficients and hence (4.10) to the differential equation for an elastic plate.

The electric potential is expressed in derivatives of $W$ by integrating (4.4.b) with respect to $\theta^3$,

$$V = -\frac{e_{(1)}^{3\gamma\delta}}{2\varepsilon_{(1)}^{33}} W|_{\gamma\delta}(\theta^3)^2 + V^*(\theta^1, \theta^2)\,, \tag{4.11}$$

where $V^*$ is an arbitrary function.

The edge conditions which have to be satisfied in order that (4.10) yields a unique solution, can be obtained from a consideration of the internal energy. For a plate in equilibrium the internal energy per unit of volume is given by (2.8). In virtue of the assumed distribution we retain from (2.8),

$$2A = T^{\alpha\beta} S_{\alpha\beta}\,. \tag{4.12}$$

By integrating (4.12) throughout the plate and using (4.3) and (4.8), the total energy $A^*$ is given by

$$2A^* = -\iint_\sigma M^{\alpha\beta} W|_{\alpha\beta}\, d\sigma\,, \tag{4.13}$$

where $d\sigma$ is an area element of the middle plane.

The internal energy consists only of a mechanical contribution. Hence we have only mechanical boundary conditions. By applying Green's theorem ([5]) and the equations of equilibrium (4.7), equation (4.13) transforms into

$$2A^* = \int_s (-M^{\alpha\beta} W|_\alpha + Q^{\beta 3} W) n_\beta\, ds - \iint_\sigma q W\, d\sigma\,. \tag{4.14}$$

Here $s$ represents the edge of the plate and $n_\beta$ the components of the outward directed normal $\boldsymbol{n}$ of unit length on $s$ in the plane of the plate. By means of partial integration and appropriate methods ([7], [9]), the known boundary conditions for a plate are obtained. In order to formulate these conditions, we introduce also a tangential vector $\boldsymbol{t}$ of unit length to $s$.

Denoting the derivatives of $W$ along $\boldsymbol{n}$ and $\boldsymbol{t}$ by $W_{,n}$, respectively $W_{,t}$; the bending moment at the edge by $M_{(t)}$ and the twisting moment by $M_{(n)}$; the derivative of $M_{(n)}$ with respect to $\boldsymbol{t}$ by $M_{(n),t}$ and the shear force at the edge by $Q$, we arrive at the following theorem.

When on every part of the edge either $M_{(t)}$ or $W_{,n}$ is given, on those parts where $M_{(n),t}$ exists either $Q + M_{(n),t}$ or $W$ and at the remaining points $W$ or the discontinuity in $M_{(n)}$, the solution of the plate equation is unique.

The derivations given in this section, remain valid when an uncharged electrode is present in an equipotential plane. Since every plane parallel to the faces is such a plane, the equations given above can also be applied to region II of an incomplete bimorph (fig. 1), since we assumed that the charge vanishes on the part of the central electrode situated in the mentioned region. Assuming that the potential on the central electrode vanishes, the function $V^*$, introduced in (4.11) vanishes in this case

## 5. Bending of an Unbounded, Complete Bimorph by Moments and Charges

An infinitely extended, complete bimorph is referred to Cartesian coordinates $(x_1, x_2, x_3)$ with $x_3 = \pm h$ defining the outermost faces. The bimorph is deformed by moments, which are uniformly distributed along the "edges" at infinity and by the action of electrical charges on the electrodes. The charge per unit area on the central electrode has a constant value $2F$. The values of the charges on the outermost electrodes are not important since they do not affect the solution inside the plate but only the external electric field. In the remainder we assume that the sum of the charges on the outermost electrodes has the opposite value of the charge on the central electrode. Hence, because they are shorted, the charge per unit area on an outermost electrode equals $-F$. We assume that the stresses, strains, electric field and electric displacement are independent of $x_1$ and $x_2$.

When the central electrode is free of charges, the solution for this problem is given in section 3.

In presence of charges additional oppositely directed, constant electric fields are generated in the plates. From (2.4) and the transition conditions (2.6) we find

$$D_3 = D_3^{(c)} \text{ sign } x_3 , \tag{5.1}$$

where

$$D_3^{(c)} = F . \tag{5.2}$$

In virtue of (2.4.a) and the presence of electrodes, $E_1$ and $E_2$ vanish. For the distribution of the remaining electrical components we assume, adding constant terms to (3.3),

$$E_3 = E_3^{(c)} \text{ sign } x_3 + E_3^{(1)} x_3 , \tag{5.3.a}$$

$$D_\alpha = D_\alpha^{(c)} \text{ sign } x_3 + D_\alpha^{(1)} x_3 . \tag{5.3.b}$$

Note that the quantities with superscript ($c$) and superscript (1) have different dimensions.

A number of components of the stress and strain tensor will also include a sign $x_3$ function and exhibit a jump at $x_3=0$. Since $T_{j3}$ and $U_j$ (and hence $S_{\alpha\beta}$) have to be continuous across the middle plane, discontinuities are only allowed to occur in $T_{\alpha\beta}$ and $S_{j3}$. Hence we assume

$$T_{\alpha\beta} = T_{\alpha\beta}^{(c)} \text{ sign } x_3 + T_{\alpha\beta}^{(1)} x_3 , \quad T_{j3} = 0 , \tag{5.4.a}$$

$$S_{\alpha\beta} = S_{\alpha\beta}^{(1)} x_3, \quad S_{j3} = S_{j3}^{(c)} \text{ sign } x_3 + S_{j3}^{(1)} x_3 . \tag{5.4.b}$$

When we substitute the terms containing variables with a superscript ($c$) into (2.1.a) and eliminate $S_{j3}^{(c)}$ we arrive at the following system of equations,

$$T_{\alpha\beta}^{(c)} = -e_{3\alpha\beta}^{(1)} E_3^{(c)} , \tag{5.5.a}$$

$$D_i^{(c)} = \varepsilon_{i3}^{(1)} E_3^{(c)} , \tag{5.5.b}$$

where $e_{3\alpha\beta}^{(1)}$ and $\varepsilon_{i3}^{(1)}$ have the same meaning as in expression (4.2), section 4.

In virtue of (5.4.a) the moments $M_{\alpha\beta}$ become

$$M_{\alpha\beta} = h^2 T_{\alpha\beta}^{(c)} + \tfrac{2}{3} h^3 T_{\alpha\beta}^{(1)} . \tag{5.6}$$

By means of (5.1), (5.2) and (5.5) $T_{\alpha\beta}^{(c)}$ can be expressed in $F$. Then we derive from (5.6),

$$T_{\alpha\beta}^{(1)} = \frac{3M_{\alpha\beta}^*}{2h^3} , \tag{5.7}$$

where

$$M_{\alpha\beta}^* = M_{\alpha\beta} + \frac{e_{3\alpha\beta}^{(1)}}{\varepsilon_{33}^{(1)}} h^2 F . \tag{5.8}$$

Hence, applying the analysis of section 3 to the coefficients with superscript (1) we find,

$$S_{ij}^{(1)} = \frac{3M_{\alpha\beta}^*}{2h^3} s_{ij\alpha\beta} . \tag{5.9.a}$$

From (2.1.b), (5.1), (5.2) and (5.5) we derive

$$S_{j3}^{(c)} = \frac{d_{3j3} - {}^E s_{j3\alpha\beta} e_{3\alpha\beta}^{(1)}}{\varepsilon_{33}^{(1)}} F . \tag{5.9.b}$$

Assuming that the conditions of definiteness (3.5) are satisfied, the mechanical displacements become,

$$U_\gamma = S_{\gamma j}^{(1)} x_j x_3 + 2 S_{\gamma 3}^{(c)} |x_3| , \tag{5.10.a}$$

$$U_3 = \tfrac{1}{2}(S_{33}^{(1)} x_3^2 - S_{\gamma b}^{(1)} x_\gamma x_b) + S_{33}^{(c)} |x_3| . \tag{5.10.b}$$

We conclude from (5.4.b) and (5.10.b) that

$$S_{\alpha\beta} = -U_{3,\alpha\beta} x_3 , \tag{5.11}$$

analogously to (3.7). The linear equations of piezoelectricity are satisfied exactly by the above-stated solution.

The potential difference $V_0 = V(-h) - V(0)$ follows from (5.1), (5.2), (5.5), (5.7) and (3.3),

$$V_0 = \frac{3\, d_{3\alpha\beta}}{4h\, {}^T\varepsilon_{33}} M^*_{\alpha\beta} - \frac{h}{\varepsilon^{(1)}_{33}} F . \tag{5.12}$$

For arbitrary bending of thin bimorphs, we assume that for the local distribution of the stresses, strains, electric field and electric displacement along the thickness, the distributions given by the exact solution for the unbounded bimorph may be used.

## 6. Approximate Equations for a Complete Bimorph, Conclusions

The bimorph is referred to the plate coordinates introduced in section 4. The plane $\theta^3 = 0$ coincides with the central electrode. The exterior faces are given by $\theta^3 = \pm h$. The bimorph is deformed by a transverse load of the form (4.6) and by electric charges on the electrodes. The potential difference between the central electrode and the shorted exterior ones is denoted by $V_0 = V(\pm h) - V(0)$ and the total charge on the central electrode by $2F^*$. Then each outermost electrode has a charge $-F^*$. We assume that either $V_0$ or $F^*$ is given.

We write for the distributions of $T^{\alpha\beta}$, $S_{\alpha\beta}$, $E_3$ and $D^i$, according to the exact solution of section 5,

$$T^{\alpha\beta} = T^{\alpha\beta}_{(c)} \operatorname{sign} \theta^3 + T^{\alpha\beta}_{(1)} \theta^3 , \tag{6.1.a}$$

$$S_{\alpha\beta} = -W|_{\alpha\beta} \theta^3 , \tag{6.1.b}$$

$$E_3 = E^{(c)}_3 \operatorname{sign} \theta^3 + E^{(1)}_3 \theta^3 , \tag{6.1.c}$$

$$D^\alpha = D^\alpha_{(c)} \operatorname{sign} \theta^3 + D^\alpha_{(1)} \theta^3 , \tag{6.1.d}$$

$$D^3 = D^3_{(c)} \operatorname{sign} \theta^3 , \tag{6.1.e}$$

where again $W = U_3(\theta^1, \theta^2, 0)$. The bending and twisting moments become

$$M^{\alpha\beta} = h^2 \left\{ T^{\alpha\beta}_{(c)} + \frac{2h}{3} T^{\alpha\beta}_{(1)} \right\} \tag{6.2}$$

and have to satisfy (4.9).

When we substitute (6.1) into (2.1.a) the following systems of equations are obtained, taking into account that $T^{j3}$ and $E_\alpha$ vanish,

$$T^{\alpha\beta}_{(c)} = -e^{3\alpha\beta}_{(1)} E^{(c)}_3 , \tag{6.3.a}$$

$$D^i_{(c)} = \varepsilon^{i3}_{(1)} E^{(c)}_3 , \tag{6.3.b}$$

$$T^{\alpha\beta}_{(1)} = -c^{\alpha\beta\gamma\delta}_{(1)} W|_{\gamma\delta} - e^{3\alpha\beta}_{(1)} E^{(1)}_3 , \tag{6.4.a}$$

$$D^\alpha_{(1)} = -e^{\alpha\gamma\delta}_{(1)} W|_{\gamma\delta} + \varepsilon^{\alpha 3}_{(1)} E^{(1)}_3 , \tag{6.4.b}$$

$$E^{(1)}_3 = \frac{e^{3\gamma\delta}_{(1)}}{\varepsilon^{33}_{(1)}} W|_{\gamma\delta} . \tag{6.4.c}$$

The coefficients $c^{\alpha\beta\gamma\delta}_{(1)}$, $e^{j\gamma\delta}_{(1)}$ and $\varepsilon^{i3}_{(1)}$ are already introduced in section 4.

Since the potential difference between the electrodes in the planes $\theta^3 = \pm h$ and $\theta^3 = 0$ is denoted by $V_0$, combination of (2.4.a) and (6.4.c) yields

$$E^{(c)}_3 = -\left\{ \frac{V_0}{h} + \frac{h e^{3\gamma\delta}_{(1)}}{2\,\varepsilon^{33}_{(1)}} W|_{\gamma\delta} \right\} . \tag{6.5}$$

By substituting (6.4.c) and (6.5) into (6.4.a) and (6.3.a) the stresses $T^{\alpha\beta}$ can be expressed in $V_0$ and derivatives of $W$,

$$T_{(c)}^{\alpha\beta} = e_{(1)}^{3\alpha\beta}\left(\frac{V_0}{h} + \frac{h e_{(1)}^{3\gamma\delta}}{2\,\varepsilon_{(1)}^{33}} W|_{\gamma\delta}\right), \tag{6.6.a}$$

$$T_{(1)}^{\alpha\beta} = -\left(c_{(1)}^{\alpha\beta\gamma\delta} + \frac{e_{(1)}^{3\alpha\beta} e_{(1)}^{3\gamma\delta}}{\varepsilon_{(1)}^{33}}\right) W|_{\gamma\delta}. \tag{6.6.b}$$

Hence the moments $M^{\alpha\beta}$ become,

$$M^{\alpha\beta} = e_{(1)}^{3\alpha\beta} h V_0 - \tfrac{2}{3} h^3 c_{(3)}^{\alpha\beta\gamma\delta} W|_{\gamma\delta}, \tag{6.7}$$

where

$$c_{(3)}^{\alpha\beta\gamma\delta} = c_{(1)}^{\alpha\beta\gamma\delta} + \frac{e_{(1)}^{3\alpha\beta} e_{(1)}^{3\gamma b}}{4\,\varepsilon_{(1)}^{33}}. \tag{6.8}$$

Substitution of (6.7) into (4.9) yields the following fourth order differential equation in $W$,

$$c_{(3)}^{\alpha\beta\gamma\delta} W|_{\alpha\beta\gamma\delta} = \frac{3q}{2h^3}. \tag{6.9}$$

Our problem is reduced to solving (6.9), under the constraints of appropriate boundary conditions.

These conditions can be derived again from a consideration of the internal energy. For a bimorph in equilibrium we obtain within the accuracy of the theory for the energy stored in the plate,

$$2A = 2\iint_\sigma \int_0^h \{-(T_{(c)}^{\alpha\beta} + T_{(1)}^{\alpha\beta}\theta^3) W|_{\alpha\beta}\theta^3 + (E_3^{(c)} + E_3^{(1)}\theta^3) D_{(c)}^3\}\, d\theta^3\, d\sigma. \tag{6.10}$$

When we carry out the integration with respect to $\theta^3$ and apply (6.2), (6.4.c) and (6.5), this expression is transformed into

$$2A = -\iint_\sigma M^{\alpha\beta} W|_{\alpha\beta}\, d\sigma - 2F^* V_0. \tag{6.11}$$

The first term in the right hand side can be treated like (4.13). Since we assumed that $V_0$ or $F^*$ is given, it is obvious that we arrive at the conditions of the theorem given in section 4.

Finally we give a relation between the total charge on the electrodes, the potential difference and the deflection. By integrating (6.3.b) over the middle plane with total area $\sigma$ and applying (6.5) we obtain

$$F^* + \frac{\varepsilon_{(1)}^{33}\sigma}{h} V_0 + \frac{h}{2}\iint_\sigma e_{(1)}^{3\gamma\delta} W|_{\gamma\delta}\, d\sigma = 0. \tag{6.12}$$

When (6.9) is solved and $V_0$ is prescribed, $F^*$ follows from (6.12). Inversely, when $F^*$ is given, $V_0$ follows from this equation.

Now region I of an incomplete bimorph (fig. 1) can be described by the approximate equations for a bimorph, given in this section and region II by the equations for a plate (section 4). The two-dimensional theory for incomplete bimorphs is completed by giving the usual conditions of continuity on the common boundary of the regions I and II and the edge conditions. The latter ones are stated in the theorem given in section 4, while at the common boundary the quantities mentioned in this theorem must be continuous.

By means of the theory presented, thin incomplete bimorphs which are statically deformed can be treated and hence also quasistatic problems.

## 7. Bending of a Circular, Incomplete, Piezoceramic Bimorph by a Concentrated Load; Experimental Data

In this section the approximate theory stated above, is applied to a circular, incomplete, piezoceramic bimorph, polarized perpendicular to the faces. The bimorph is described by cylindrical coordinates $(r, \varphi, z)$, with the $z$-axis along the axis of rotatory symmetry. The central electrode is given by $z=0$ and the outer faces by $z=\pm h$. The latter ones are partly covered by electrodes, occupying the region $0 \leqq r \leqq a$. The edge of the bimorph is given by $r=b$. The bimorph is simply supported along its edge and loaded by a concentrated force $P$ acting along the positive $z$-axis (fig. 3). The potential difference $V_0$ between the inner electrode and the exterior ones is obtained as a function of $a/b$ in case the outer electrodes are insulated from the central one and compared with experimental data.

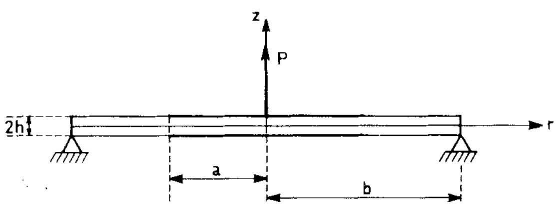


Figure 3. Cross section of the circular bimorph, loaded by a concentrated force.

With respect to Cartesian coordinates $(x_1, x_2, x_3)$ with the $x_3$-axis along the direction of polarization, the elastic, piezoelectric and dielectric constants for a piezoceramic may conveniently be represented in the following tables ([1]),

$$
\begin{array}{cccccc}
{}^E s_{1111} & {}^E s_{1122} & {}^E s_{1133} & 0 & 0 & 0 \\
 & {}^E s_{1111} & {}^E s_{1133} & 0 & 0 & 0 \\
 & & {}^E s_{3333} & 0 & 0 & 0 \\
 & & & {}^E s_{1313} & 0 & 0 \\
 & & & & {}^E s_{1313} & 0 \\
 & & & & & {}^E s_{1212}
\end{array}
$$

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & d_{113} & 0 \\
0 & 0 & 0 & d_{113} & 0 & 0 \\
d_{311} & d_{311} & d_{333} & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{ccc}
{}^T\varepsilon_{11} & 0 & 0 \\
 & {}^T\varepsilon_{11} & 0 \\
 & & {}^T\varepsilon_{33}
\end{array}
$$

where ${}^E s_{1212} = \frac{1}{2}({}^E s_{1111} - {}^E s_{1122})$. Since we neglected $T_{j3}$, $j=1, 2, 3$, and $E_\alpha$, $\alpha=1, 2$, we retain from (2.1) with respect to $(x_1, x_2, x_3)$,

$$
T^{11} = \frac{Y}{1-\nu^2}\{S_{11} + \nu S_{22} - (1+\nu) d_{311} E_3\}, \tag{7.1.a}
$$

$$
T^{22} = \frac{Y}{1-\nu^2}\{S_{22} + \nu S_{11} - (1+\nu) d_{311} E_3\}, \tag{7.1.b}
$$

$$D^3 = \frac{Yd_{311}}{1-\nu}(S_{11}+S_{22}) + {}^T\varepsilon_{33}\left(1-\frac{2k_1}{1-\nu}\right)E_3 . \tag{7.1.c}$$

Here $Y$ represents the Young's modulus for tension/compression parallel to the plane of the plate, $Y=({}^E s_{1111})^{-1}$, and $\nu$ Poisson's ratio, $\nu=-Y^E s_{1122}$. The constant $k_1$ is the square of an electromechanical coupling factor ([1]),

$$k_1 = \frac{Y(d_{311})^2}{{}^T\varepsilon_{33}} . \tag{7.2}$$

From (4.2) and (7.1) we conclude that with respect to Cartesian coordinates,

$$c_{(1)}^{1111} = c_{(1)}^{2222} = \frac{Y}{1-\nu^2} , \tag{7.3.a}$$

$$c_{(1)}^{1122} = \frac{Y\nu}{1-\nu^2} , \tag{7.3.b}$$

$$e_{(1)}^{311} = e_{(1)}^{322} = \frac{Yd_{311}}{1-\nu} , \tag{7.3.c}$$

$$\varepsilon_{(1)}^{33} = {}^T\varepsilon_{33}\left(1-\frac{2k_1}{1-\nu}\right) . \tag{7.3.d}$$

Hence the coefficients $c_{(2)}^{1111}$, $c_{(2)}^{2222}$ and $c_{(2)}^{1122}$, defined by (4.5), become

$$c_{(2)}^{1111} = c_{(2)}^{2222} = \frac{Y(1+k_2)}{1-\nu^2} , \tag{7.4.a}$$

$$c_{(2)}^{1122} = \frac{Y(\nu+k_2)}{1-\nu^2} , \tag{7.4.b}$$

and $c_{(3)}^{1111}$, $c_{(3)}^{2222}$, $c_{(3)}^{1122}$, given in (6.8),

$$c_{(3)}^{1111} = c_{(3)}^{2222} = \frac{Y\left(1+\dfrac{k_2}{4}\right)}{1-\nu^2} , \tag{7.5.a}$$

$$c_{(3)}^{1122} = \frac{Y\left(\nu+\dfrac{k_2}{4}\right)}{1-\nu^2} . \tag{7.5.b}$$

The constant $k_2$ is related to $\nu$ and $k_1$ by

$$k_2 = \frac{k_1(1+\nu)}{1-\nu-2k_1} . \tag{7.6}$$

Indicating the elastic, piezoelectric and dielectric coefficients with respect to $r$, $\varphi$ and $z$ by a bar, we obtain

$$\bar{c}_{(m)}^{1111} = c_{(m)}^{1111}, \quad \bar{c}_{(m)}^{1122} = r^{-2} c_{(m)}^{1122}, \quad \bar{c}_{(m)}^{2222} = r^{-4} c_{(m)}^{1111} , \tag{7.7.a}$$

$$\bar{e}_{(1)}^{311} = e_{(1)}^{311}, \quad \bar{e}_{(1)}^{322} = r^{-2} e_{(1)}^{311}, \quad \bar{\varepsilon}_{(1)}^{33} = r^{-2} \varepsilon_{(1)}^{33} , \tag{7.7.b}$$

where $m$ is 2 or 3. Since in the coordinates $(r, \varphi, z)$

$$W|_{11} = W_{,11}, \quad W|_{22} = rW_{,1} , \tag{7.8}$$

the moments $M^{11}$ and $r^2 M^{22}$ for region II become, using (4.4), (4.8), (7.4) and (7.8),

$$M^{11} = -\frac{2h^3 Y}{3(1-\nu^2)}\{(1+k_2)W_{,11} + (\nu + k_2)r^{-1} W_{,1}\} \tag{7.9.a}$$

$$r^2 M^{22} = -\frac{2h^3 Y}{3(1-\nu^2)}\{(\nu + k_2)W_{,11} + (1+k_2)r^{-1} W_{,1}\}\,. \tag{7.9.b}$$

Similarly we have in region I

$$M^{11} = \frac{Yd_{311} h V_0}{1-\nu} - \frac{2h^3 Y}{3(1-\nu^2)}\left\{\left(1 + \frac{k_2}{4}\right) W_{,11} + \left(\nu + \frac{k_2}{4}\right) r^{-1} W_{,1}\right\}, \tag{7.10.a}$$

$$r^2 M^{22} = \frac{Yd_{311} h V_0}{1-\nu} - \frac{2h^3 Y}{3(1-\nu^2)}\left\{\left(\nu + \frac{k_2}{4}\right) W_{,11} + \left(1 + \frac{k_2}{4}\right) r^{-1} W_{,1}\,. \right. \tag{7.10.b}$$

The equation of equilibrium for both regions follows from (4.7) and reads,

$$M^{11}{}_{,1} + r^{-1} M^{11} - r M^{22} = -\frac{P}{2\pi r}\,. \tag{7.11}$$

Substitution of (7.9), respectively (7.10) into (7.11) yields

$$W_{,111} + r^{-1} W_{,11} - r^{-2} W_{,1} = \frac{P}{2\pi r K} \tag{7.12}$$

for region II and

$$W_{,111} + r^{-1} W_{,11} - r^{-2} W_{,1} = \frac{P}{2\pi r K^*} \tag{7.13}$$

for region I. The constants $K$ and $K^*$ are given by

$$K = \frac{2h^3 Y(1+k_2)}{3(1-\nu^2)}\,, \tag{7.14.a}$$

$$K^* = \frac{2h^3 Y\left(1 + \frac{k_2}{4}\right)}{3(1-\nu^2)}\,. \tag{7.14.b}$$

The equations (7.12) and (7.13) have the general solution ([10]),

$$W = \frac{Pb^2}{8\pi K}\{(r')^2 \log r' + C_1 (r')^2 + C_2 \log r' + C_3\}\,, \tag{7.15}$$

respectively

$$W = \frac{Pb^2}{8\pi K^*}\{(r')^2 \log r' + C_1^* (r')^2 + C_2^* \log r' + C_3^*\}\,, \tag{7.16}$$

where $r' = r/b$. The constants $C_1 \dots C_3^*$ and the potential difference $V_0$ are obtained from the conditions that $M^{11}$ and $W$ vanish for $r'=1$, that $M^{11}$, $W$ and $W_{,1}$ are continuous for $r' = a/b$, that $W$ is bounded for $r'=0$ and from the equation

$$V_0 = \frac{-h^2 Yd_{311}}{{}^T\varepsilon_{33}(1-\nu-2k_1)} W_{,1} \quad (r=a)\,, \tag{7.17}$$

which is derived from (6.12). Elimination of $C_1 \dots C_3^*$ from these conditions and from (7.17) yields

$$V_0 = \frac{Yd_{311} h^2 P}{4\pi K\, {}^T\varepsilon_{33}(1-\nu-2k_1)}\left(-\log\frac{a}{b} + \frac{1+k_2}{1+\nu+2k_2}\right). \tag{7.18}$$

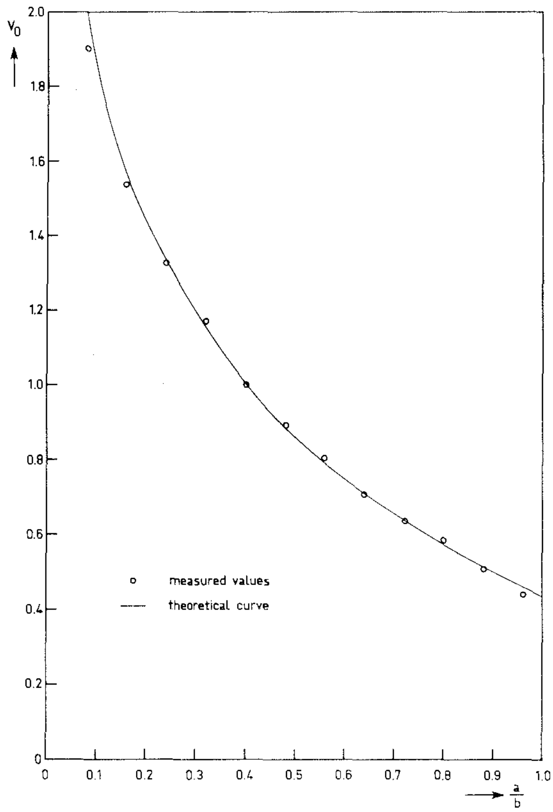


Figure 4. Experimental values and theoretical curve for $V_0$.

This result is checked experimently. A detailed description of the experiment is given in [11]. In fig. 4 the measured value of the potential difference $V_0$ for a number of values of $a/b$ is plotted. Since the coefficients of the material and the applied load were not known exactly, some suitable measurements are used for determining the coefficient of the logarithmic function and the constant in (7.18). By applying these values, the theoretical curve (7.18) is plotted. The remaining measured values are in good agreement with the theory.

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